

FERMI'S TRICK AND SYMPLECTIC CAPACITIES: A GEOMETRIC PICTURE OF QUANTUM STATES

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Abstract

We extend the notion of “quantum blob” studied in previous work to excited states of the generalized harmonic oscillator in n dimensions. This extension is made possible by Fermi’s observation in 1930 that the state of a quantum system may be defined in two different (but equivalent) ways, namely by its wavefunction Ψ or by a certain function g_F on phase space canonically associated with Ψ . We study Fermi’s function when Ψ is a Gaussian (generalized coherent state). A striking result is that we can use the Ekeland–Hofer symplectic capacities to characterize the Fermi functions of the excited states of the generalized harmonic oscillator, leading to new insight on the relationship between symplectic topology and quantum mechanics.

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1 Introduction

1.1 What We Want to Do

We address the question whether it is possible to represent geometrically a function ψ of the variables $x = (x_1, x_2, \dots, x_n)$. The problem is in fact easy to answer if ψ is a Gaussian function because then its Wigner transform is proportional to a Gaussian $e^{-\frac{1}{\hbar} S^T S z \cdot z}$ where S is a symplectic matrix uniquely determined by ψ . It follows that there is a one-to-one correspondence between Gaussians and the sets $S^T S z \cdot z \leq \hbar$. We have called these sets “quantum blobs” in [12, 13, 14, 15, 16, 17, 18]; the interest of these quantum blobs comes from the fact that they represent minimum uncertainty sets in phase space.

The Gaussian function

$$\Psi_0(x) = e^{-x^2/2\hbar}, \quad (1)$$

is the (unnormalized) ground state of the one-dimensional harmonic oscillator with mass and frequency equal to one: $\hat{H}\Psi_0 = E_0\Psi_0$ where $E_0 = \frac{1}{2}\hbar$ and

$$\hat{H} = \frac{1}{2} \left(-\hbar^2 \frac{d^2}{dx^2} + x^2 \right) \quad (2)$$

This operator is the quantization of the classical oscillator Hamiltonian

$$H(x, p) = \frac{1}{2}(p^2 + x^2) \quad (3)$$

The set Ω_0 defined by the inequality $H \leq E_0$ is the interior of the energy hypersurface $H \leq E_0$; it is the disk $p^2 + x^2 \leq \hbar$ with radius $R_0 = \sqrt{\hbar}$. Let us now consider the N -th excited state of the operator \hat{H} ; it is the (unnormalized) Hermite function

$$\Psi_N(x) = e^{-x^2/2\hbar} H_N(x/\sqrt{\hbar}) \quad (4)$$

where

$$H_N(x) = (-1)^n e^{x^2} \frac{d^N}{dx^N} e^{-x^2} \quad (5)$$

is the N -th Hermite polynomial. It is a solution of $\hat{H}\Psi_N = (N + \frac{1}{2})\hbar\Psi_N$ and the set Ω_N defined by the inequality $H \leq E_N = (2N + 1)\hbar$ is again a disk, but this time with radius $R_N = \sqrt{(N + \frac{1}{2})\hbar}$.

In this paper we introduce a non-trivial extension of the notion of “quantum blob” we defined and studied in previous work. Quantum blobs are

deformations of the phase space ball $|x|^2 + |p|^2 \leq \hbar$ by translations and linear canonical transformations. Their interest come from the fact that they provide us with a coarse-graining of phase space different from the usual coarse graining by cubes with volume $\sim \hbar^n$ commonly used in statistical mechanics. They appear as space units of minimum uncertainty in one-to-one correspondence with the generalized coherent states familiar from quantum optics, and have allowed us to recover the exact ground states of generalized harmonic oscillators, as well as the semiclassical energy levels of quantum systems with completely integrable Hamiltonian function, and to explain them in terms of the topological notion of symplectic capacity [24, 30] originating in Gromov's [23] non-squeezing theorem (alias "the principle of the symplectic camel"). Quantum blobs, do not, however, allow a characterization of excited states; for instance there is no obvious relation between them and the Hermite functions. Why this does not work is easy to understand: quantum blobs correspond to the states saturating the Schrödinger–Robertson inequalities

$$(\Delta X_j)^2 (\Delta P_j)^2 \geq \Delta(X_j, P_j)^2 + \frac{1}{4} \hbar^2, \quad 1 \leq j \leq n; \quad (6)$$

as is well-known [21] the quantum states for which all these inequalities become equalities are Gaussians, in this case precisely those who are themselves the ground states of generalized harmonic oscillators. As soon as one consider the excited states the corresponding eigenfunctions are Hermite functions and for these the inequalities (6) are strict. The way out of this difficulty is to define new phase space objects, the "Fermi blobs" of the title of this paper. Such an approach should certainly be welcome in times where phase space is beginning to be taken seriously (see the recent review paper [7]).

1.2 How We Will Do It

We will show that a complete geometric picture of excited states can be given using an idea of the physicist Enrico Fermi in a largely forgotten paper [8] from 1930. Fermi associates to every quantum state Ψ a certain hypersurface $g_F(x, p) = 0$ in phase space. The underlying idea is actually surprisingly simple. It consists in observing that any complex twice continuously differentiable function $\Psi(x) = R(x)e^{i\Phi(x)/\hbar}$ ($R(x) \geq 0$ and $\Phi(x)$ real) defined on \mathbb{R}^n satisfies the partial differential equation

$$\left[(-i\hbar \nabla_x - \nabla_x \Phi)^2 + \hbar^2 \frac{\nabla_x^2 R}{R} \right] \Psi = 0. \quad (7)$$

where ∇_x^2 is the Laplace operator in the variables x_1, \dots, x_n (it is assumed that $R(x) \neq 0$ for x in some subset of \mathbb{R}^n). Performing the gauge transformation $-i\hbar\nabla_x \rightarrow -i\hbar\nabla_x - \nabla_x\Phi$, this equation is in fact equivalent to the trivial equation

$$\left(-\hbar^2\nabla_x^2 + \hbar^2\frac{\nabla_x^2 R}{R}\right)R = 0. \quad (8)$$

The operator

$$\widehat{g_F} = (-i\hbar\nabla_x - \nabla_x\Phi)^2 + \hbar^2\frac{\nabla_x^2 R}{R} \quad (9)$$

appearing in the left-hand side of Eqn. (7) is the quantisation (in every reasonable physical quantisation scheme) of the real observable

$$g_F(x, p) = (p - \nabla_x\Phi)^2 + \hbar^2\frac{\nabla_x^2 R}{R} \quad (10)$$

and the equation $g_F(x, p) = 0$ in general determines a hypersurface \mathcal{H}_F in phase space $\mathbb{R}_{x,p}^{2n}$ which Fermi ultimately *identifies* with the state Ψ itself. The remarkable thing with this construction is that it shows that to an arbitrary function Ψ it associates a Hamiltonian function of the classical type

$$H = (p - \nabla_x\Phi)^2 + V \quad (11)$$

even if Ψ is the solution of another partial (or pseudo-differential) equation. We notice that when Ψ is an eigenstate of the operator $\widehat{H}\Psi = E\Psi$ then $g_F = H - E$ and \mathcal{H}_F is just the energy hypersurface $H(x, p) = E$.

Of course, Fermi's analysis was very heuristic and its mathematical rigour borders the unacceptable (at least by modern standards). Fermi's paper has recently been rediscovered by Benenti [2] and Benenti and Strini [3], who study its relationship with the level sets of the Wigner transform of Ψ .

Notation 1 *The points in configuration and momentum space are written $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$ respectively; in formulas x and p are viewed as column vectors. We will also use the collective notation $z = (x, p)$ for the phase space variable. The matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (0 and I the $n \times n$ zero and identity matrices) defines the standard symplectic form on the phase space \mathbb{R}_x^{2n} via the formula $\sigma(z, z') = Jz \cdot z' = p \cdot x' - p' \cdot x$. We write $\hbar = h/2\pi$, h being Planck's constant. The symplectic group is denoted by $\text{Sp}(2n, \mathbb{R})$: it is the multiplicative group of all real $2n \times 2n$ matrices S such that $\sigma(Sz, Sz') = \sigma(z, z')$ for all z, z' .*

2 Symplectic Capacities and Quantum Blobs

To generalize the discussion above to the multi-dimensional case we have to introduce some concepts from symplectic topology. For a review of these notions see de Gosson and Luef [22].

2.1 Symplectic Capacities

Intrinsic symplectic capacities

An *intrinsic* symplectic capacity assigns a non-negative number (or $+\infty$) $c(\Omega)$ to every subset Ω of phase space \mathbb{R}^{2n} ; this assignment is subjected to the following properties:

- **Monotonicity:** If $\Omega \subset \Omega'$ then $c(\Omega) \leq c(\Omega')$;
- **Symplectic invariance:** If f is a canonical transformation (linear, or not) then $c(f(\Omega)) = c(\Omega)$;
- **Conformality:** If λ is a real number then $c(\lambda\Omega) = \lambda^2 c(\Omega)$; here $\lambda\Omega$ is the set of all points λz when $z \in \Omega$;
- **Normalization:** We have

$$c(B^{2n}(R)) = \pi R^2 = c(Z_j^{2n}(R)); \quad (12)$$

here $B^{2n}(R)$ is the phase-space ball $|x|^2 + |p|^2 \leq R^2$ and $Z_j^{2n}(R)$ the phase-space cylinder $x_j^2 + p_j^2 \leq R^2$.

Let c be a symplectic capacity on the phase plane \mathbb{R}^2 . We have $c(\Omega) = \text{Area}(\Omega)$ when Ω is a connected and simply connected surface. In the general case there exist infinitely many intrinsic symplectic capacities, but they all agree on phase space ellipsoids as we will see below. The smallest symplectic capacity is denoted by c_{\min} (“Gromov width”): by definition $c_{\min}(\Omega)$ is the supremum of all numbers πR^2 such that there exists a canonical transformation such that $f(B^{2n}(R)) \subset \Omega$. The fact that c_{\min} really is a symplectic capacity follows from a deep and difficult topological result, Gromov’s [23] symplectic non-squeezing theorem, alias the principle of the symplectic camel. (For a discussion of Gromov’s theorem from the point of view of Physics see de Gosson [17], de Gosson and Luef [22].) Another useful example is provided by the Hofer–Zehnder [24] capacity c^{HZ} . It has the property

that it is given by the integral of the action form $pdx = p_1dx_1 + \dots + p_ndx_n$ along a certain curve:

$$c^{\text{HZ}}(\Omega) = \oint_{\gamma_{\min}} pdx \quad (13)$$

when Ω is a compact convex set in phase space; here γ_{\min} is the shortest (positively oriented) Hamiltonian periodic orbit carried by the boundary $\partial\Omega$ of Ω . This formula agrees with the usual notion of area in the case $n = 1$.

It turns out that all intrinsic symplectic capacities agree on phase space ellipsoids, and are calculated as follows (see e.g. [16, 22, 24]). Let M be a $2n \times 2n$ positive-definite matrix M and consider the ellipsoid:

$$\Omega_{M,z_0} : M(z - z_0)^2 \leq 1. \quad (14)$$

Then, for every intrinsic symplectic capacity c we have

$$c(\Omega_{M,z_0}) = \pi/\lambda_{\max}^{\sigma} \quad (15)$$

where λ_{\max}^{σ} is the largest symplectic eigenvalue of M . The symplectic eigenvalues of a positive definite matrix are defined as follows: the matrix JM (J the standard symplectic matrix) is equivalent to the antisymmetric matrix $M^{1/2}JM^{1/2}$ hence its $2n$ eigenvalues are of the type $\pm i\lambda_1^{\sigma}, \dots, \pm i\lambda_n^{\sigma}$ where $\lambda_j^{\sigma} > 0$. The positive numbers $\lambda_1^{\sigma}, \dots, \lambda_n^{\sigma}$ are called the *symplectic eigenvalues* of the matrix M .

In particular, if X and Y are real symmetric $n \times n$ matrices, then the symplectic capacity of the ellipsoid

$$\Omega_{(A,B)} : Xx^2 + Yp^2 \leq 1 \quad (16)$$

is given by

$$c(\Omega_{(A,B)}) = \pi/\sqrt{\lambda_{\max}} \quad (17)$$

where λ_{\max} is the largest eigenvalue of AB .

Extrinsic symplectic capacities

The definition of an extrinsic symplectic capacity is similar to that of an intrinsic capacity, but one weakens the normalization condition (12) by only requiring that:

- **Nontriviality:** $c(B^{2n}(R)) < +\infty$ and $c(Z_j^{2n}(R)) < +\infty$.

In [6] Ekeland and Hofer defined a sequence $c_1^{\text{EH}}, c_2^{\text{EH}}, \dots, c_k^{\text{EH}}, \dots$ of extrinsic symplectic capacities satisfying the nontriviality properties

$$c_k^{\text{EH}}(B^{2n}(R)) = \left\lfloor \frac{k+n-1}{n} \right\rfloor \pi R^2, \quad c_k^{\text{EH}}(Z_j^{2n}(R)) = k\pi R^2. \quad (18)$$

Of course c_1^{EH} is an intrinsic capacity; in fact it coincides with the Hofer–Zehnder capacity on bounded convex sets Ω . We have

$$c_1^{\text{EH}}(\Omega) \leq c_2^{\text{EH}}(\Omega) \leq \dots \leq c_k^{\text{EH}}(\Omega) \leq \dots \quad (19)$$

The Ekeland–Hofer capacities have the property that for each k there exists an integer $N \geq 0$ and a closed characteristic γ of $\partial\Omega$ such that

$$c_k^{\text{EH}}(\Omega) = N \left| \oint_{\gamma} p dx \right| \quad (20)$$

(in other words, $c_k^{\text{EH}}(\Omega)$ is a value of the *action spectrum* [5] of the boundary $\partial\Omega$ of Ω); this formula shows that $c_k^{\text{EH}}(\Omega)$ is solely determined by $\partial\Omega$; therefore the notation $c_k^{\text{EH}}(\partial\Omega)$ is often used in the literature. The Ekeland–Hofer capacities c_k^{EH} allow us to classify phase-space ellipsoids. In fact, the non-decreasing sequence of numbers $c_k^{\text{EH}}(\Omega_M)$ is determined as follows for an ellipsoid $\Omega : Mz \cdot z \leq 1$ (M symmetric and positive-definite): let $(\lambda_1^\sigma, \dots, \lambda_n^\sigma)$ be the symplectic eigenvalues of M ; then

$$\{c_k^{\text{EH}}(\Omega) : k = 1, 2, \dots\} = \{N\pi\lambda_j^\sigma : j = 1, \dots, n; N = 0, 1, 2, \dots\}. \quad (21)$$

Equivalently, the increasing sequence $c_1^{\text{EH}}(\Omega) \leq c_2^{\text{EH}}(\Omega) \leq \dots$ is obtained by writing the numbers $N\pi\lambda_j^\sigma$ in increasing order with repetitions if a number occurs more than once.

2.2 Quantum Blobs

By definition a quantum blob $\mathcal{QB}^{2n}(z_0, S)$ is the image of the phase space ball $B^{2n}(S^{-1}z_0, \sqrt{\hbar}) : |z - S^{-1}z_0| \leq \sqrt{\hbar}$ by a linear canonical transformation (identified with a symplectic matrix S). A quantum blob is thus a phase space ellipsoid with symplectic capacity $\pi\hbar = \frac{1}{2}h$, but it is not true that, conversely, an arbitrary phase space ellipsoid with symplectic capacity $\frac{1}{2}h$ is a quantum blob. One can however show (de Gosson [14, 15, 16], de Gosson and Luef [22]) that such an ellipsoid contains a unique quantum blob. One proves (ibid.) that a quantum blob $\mathcal{QB}^{2n}(z_0, S)$ is characterized by the two following *equivalent* properties:

- The intersection of the ellipsoid $\mathcal{QB}^{2n}(z_0, S)$ with a plane passing through z_0 and parallel to any of the plane of canonically conjugate coordinates x_j, p_j in \mathbb{R}_z^{2n} is an ellipse with area $\frac{1}{2}\hbar$;
- The supremum of the set of all numbers πR^2 such that the ball $B^{2n}(\sqrt{R}) : |z| \leq R$ can be embedded into $\mathcal{QB}^{2n}(z_0, S)$ using canonical transformations (linear, or not) is $\frac{1}{2}\hbar$. Hence no phase space ball with radius $R > \sqrt{\hbar}$ can be “squeezed” inside $\mathcal{QB}^{2n}(z_0, S)$ using only canonical transformations.

It turns out (de Gosson [16]) that in the first of these conditions one can replace the plane of conjugate coordinates with any symplectic plane (a symplectic plane is a two-dimensional subspace of \mathbb{R}_z^{2n} on which the restriction of the symplectic form σ is again a symplectic form). There is a natural action

$$\mathrm{Sp}(2n, \mathbb{R}) \times \mathcal{QB}(2n, \mathbb{R}) \longrightarrow \mathcal{QB}(2n, \mathbb{R})$$

of the symplectic group on quantum blobs.

3 Generalized Coherent States

3.1 The Fermi Function of a Gaussian

We next consider arbitrary (normalized) generalized coherent states

$$\Psi_{X,Y}(x) = \left(\frac{1}{\pi\hbar} \right)^{n/4} (\det X)^{1/4} \exp \left[-\frac{1}{2\hbar} (X + iY)x \cdot x \right] \quad (22)$$

where X and Y are real symmetric $n \times n$ matrices, and X is positive definite. Setting $\Phi(x) = -\frac{1}{2}Yx \cdot x$ and $R(x) = \exp \left(-\frac{1}{2\hbar}Xx \cdot x \right)$ we have

$$\nabla_x \Phi(x) = -Yx \quad , \quad \frac{\nabla_x^2 R(x)}{R(x)} = -\frac{1}{\hbar} \mathrm{Tr} X + \frac{1}{\hbar^2} X^2 x \cdot x \quad (23)$$

hence the Fermi function of $\Psi_{X,Y}$ is the quadratic form

$$g_F(x, p) = (p + Yx)^2 + X^2 x \cdot x - \hbar \mathrm{Tr} X. \quad (24)$$

We can rewrite this formula as

$$g_F(x, p) = M_F z \cdot z - \hbar \mathrm{Tr} X \quad (25)$$

($z = (x, p)$) where M_F is the symmetric matrix

$$M_F = \begin{pmatrix} X^2 + Y^2 & Y \\ Y & I \end{pmatrix}. \quad (26)$$

A straightforward calculation shows that we have the factorization

$$M_F = S^T \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} S \quad (27)$$

where S is the *symplectic* matrix

$$S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix}. \quad (28)$$

It turns out –and this is really a striking fact!– that M_F is closely related to the Wigner transform

$$W\Psi_{X,Y}(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \Psi_{X,Y}(x + \frac{1}{2}y) \Psi_{X,Y}^*(x - \frac{1}{2}y) dy \quad (29)$$

of the state $\Psi_{X,Y}$ because we have

$$W\Psi_{X,Y}(z) = \left(\frac{1}{\pi\hbar}\right)^n \exp\left(-\frac{1}{\hbar}Gz \cdot z\right) \quad (30)$$

where G is the symplectic matrix

$$G = S^T S = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix} \quad (31)$$

(see e.g. [16, 27]). When $n = 1$ and $\Psi_{X,Y}(x) = \Psi_0(x)$ the fiducial coherent state (1) we have $S^{-1}D^{-1/2}S = I$ and $\text{Tr } X = 1$ hence the formula

$$W\Psi_0(z) = \left(\frac{1}{\pi\hbar}\right)^{1/4} \frac{1}{e} \exp\left[-\frac{1}{\hbar}M_F z \cdot z\right]$$

already observed by Benenti and Strini in [3].

3.2 Geometric Interpretation

Recall (formula (15)) that the symplectic capacity $c(\Omega)$ of an ellipsoid $Mz \cdot z \leq 1$ (M a symmetric positive definite $2n \times 2n$ matrix) is given by

$$c(\Omega) = \pi/\lambda_{\max}^\sigma \quad (32)$$

where $\lambda_{\max}^\sigma = \max\{\lambda_1^\sigma, \dots, \lambda_n^\sigma\}$, the λ_j^σ being the symplectic eigenvalues of M . We denote by Ω_F the phase space ellipsoid defined by $g_F(x, p) \leq 0$, that is:

$$\Omega_F : M_F z \cdot z \leq \hbar \operatorname{Tr} X;$$

it is the ellipsoid bounded by the Fermi hypersurface \mathcal{H}_F corresponding to the generalized coherent state $\Psi_{X,Y}$. Let us perform the symplectic change of variables $z' = Sz$; in the new coordinates the ellipsoid Ω_F is represented by the inequality

$$Xx' \cdot x' + Xp' \cdot p' \leq \hbar \operatorname{Tr} X \quad (33)$$

hence $c(\Omega_F)$ equals the symplectic capacity of the ellipsoid (33). Applying the rule above we thus have to find the symplectic eigenvalues of the block-diagonal matrix $\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$; a straightforward calculation shows that these are just the eigenvalues $\omega_1, \dots, \omega_n$ of X and hence

$$c(\Omega_F) = \pi \hbar \operatorname{Tr} X / \omega_{\max} \quad (34)$$

where $\omega_{\max} = \max\{\omega_1, \dots, \omega_n\}$. In view of the trivial inequality

$$\omega_{\max} \leq \operatorname{Tr} X = \sum_{j=1}^n \omega_j \leq n \omega_{\max} \quad (35)$$

we have

$$\frac{1}{2}h \leq c(\Omega_F) \leq \frac{nh}{2}. \quad (36)$$

An immediate consequence of the inequality $\frac{1}{2}h \leq c(\Omega_F)$ is that the Fermi ellipsoid Ω_F of a generalized coherent state always contains a quantum blob; this is of course consistent with the uncertainty principle.

Notice that when all the eigenvalues ω_j are equal to a number ω then $c(\Omega_F) = nh/2$; in particular when $n = 1$ we have $c(\Omega_F) = h/2$ which is exactly the action calculated along the trajectory corresponding to the ground state. This observation leads us to the following question: what is the precise geometric meaning of formula (34)? Let us come back to the interpretation of the ellipsoid defined by the inequality (33). We have seen that the symplectic eigenvalues of the matrix $\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ are precisely the eigenvalues ω_j , $1 \leq j \leq n$, of the positive-definite matrix X . It follows that there exist linear symplectic coordinates (x'', p'') in which the equation of the ellipsoid Ω_F takes the normal form

$$\sum_{j=1}^n \omega_j (x_j''^2 + p_j''^2) \leq \sum_{j=1}^n \hbar \omega_j \quad (37)$$

whose quantum-mechanical interpretation is clear: dividing both sides by two we get the energy shell of the anisotropic harmonic oscillator in its ground state. Consider now the planes $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ of conjugate coordinates $(x_1, p_1), (x_2, p_2), \dots, (x_n, p_n)$. The intersection of the ellipsoid Ω_F with these planes are the circles

$$\begin{aligned} C_1 : \omega_1(x_1''^2 + p_1''^2) &\leq \sum_{j=1}^n \hbar \omega_j \\ C_2 : \omega_2(x_2''^2 + p_2''^2) &\leq \sum_{j=1}^n \hbar \omega_j \\ &\dots\dots\dots \\ C_n : \omega_n(x_n''^2 + p_n''^2) &\leq \sum_{j=1}^n \hbar \omega_j. \end{aligned}$$

Formula (34) says that $c(\Omega_F)$ is the area of the circle C_j with smallest radius, and this corresponds to the index j such that $\omega_j = \omega_{\max}$. This is of course perfectly in accordance with the definition of the Hofer–Zehnder capacity $c^{\text{HZ}}(\Omega_F)$ since all symplectic capacities agree on ellipsoids. We are now led to another question: is there any way to describe topologically Fermi’s ellipsoid in such a way that the areas of every circle C_j becomes apparent? The problem with the standard capacity of an ellipsoid is that it only “sees” the smallest cut of that ellipsoid by a plane of conjugate coordinate. The way out of this difficult lies in the use of the Ekeland–Hofer capacities c_j^{EH} discussed above. To illustrate the idea, let us first consider the case $n = 2$; it is no restriction to assume $\omega_1 \leq \omega_2$. If $\omega_1 = \omega_2$ then the ellipsoid

$$\omega_1(x_1''^2 + p_1''^2) + \omega_2(x_2''^2 + p_2''^2) \leq \hbar \omega_1 + \hbar \omega_2 \quad (38)$$

is just the ball $B^2(\sqrt{2\hbar})$ whose symplectic capacity is $2\pi\hbar = h$. Suppose now that $\omega_1 < \omega_2$. Then the Ekeland–Hofer capacities are the numbers

$$\frac{\pi\hbar}{\omega_2}(\omega_1 + \omega_2), \frac{\pi\hbar}{\omega_1}(\omega_1 + \omega_2), \frac{2\pi\hbar}{\omega_2}(\omega_1 + \omega_2), \frac{2\pi\hbar}{\omega_1}(\omega_1 + \omega_2), \dots \quad (39)$$

and hence

$$c_1^{\text{EH}}(\Omega_F) = c(\Omega_F) = \frac{\pi\hbar}{\omega_2}(\omega_1 + \omega_2).$$

What about $c_2^{\text{EH}}(\Omega_F)$? A first glance at the sequence (39) suggests that we have

$$c_2^{\text{EH}}(\Omega_F) = \frac{\pi\hbar}{\omega_1}(\omega_1 + \omega_2)$$

but this is only true if $\omega_1 < \omega_2 \leq 2\omega_1$ because if $2\omega_1 < \omega_2$ then $(\omega_1 + \omega_2)/\omega_2 < (\omega_1 + \omega_2)/\omega_1$ so that in this case

$$c_2^{\text{EH}}(\Omega_{\text{F}}) = \frac{\pi\hbar}{\omega_2}(\omega_1 + \omega_2) = c_1^{\text{EH}}(\Omega_{\text{F}}).$$

The Ekeland–Hofer capacities thus allow a geometrical classification of the eigenstates.

4 Fermi Function and Excited States

The generalized coherent states can be viewed as the ground states of a generalized harmonic oscillator, with Hamiltonian function a homogeneous quadratic polynomial in the position and momentum coordinates:

$$H(x, p) = \sum_{i,j} a_{ij} p_i p_j + b_{ij} p_i x_j + c_{ij} x_i x_j.$$

Such a function can always be put in the form

$$H(z) = \frac{1}{2} M z \cdot z \quad (40)$$

where M is a symmetric matrix (the Hessian matrix, i.e. the matrix of second derivatives, of H). We will assume for simplicity that M is positive-definite; we can then always bring it into the normal form

$$K(z) = \sum_{j=1}^n \frac{\omega_j}{2} (x_j^2 + p_j^2)$$

using a linear symplectic transformation of the coordinates (symplectic diagonalization): there exists a symplectic matrix S (depending on M) such that

$$S^T M S = D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \quad (41)$$

where Λ is a diagonal matrix whose diagonal entries consist of the symplectic spectrum $\omega_1, \dots, \omega_n$ of M . Thus, we have $K(z) = H(Sz)$, or, equivalently,

$$H(z) = K(S^{-1}z) \quad (42)$$

The ground state of each one-dimensional quantum oscillator

$$\hat{K}_j = \frac{\omega_j}{2} \left(x_j^2 - \hbar^2 \frac{\partial^2}{\partial x_j^2} \right)$$

is the solution of $\hat{K}_j \Psi = \frac{1}{2} \hbar \omega_j \Psi$, it is thus the one-dimensional fiducial coherent state $(\pi \hbar)^{-1/4} e^{-x^2/2\hbar}$. It follows that the ground Ψ_0 state of $\hat{K} = \sum_j \hat{K}_j$ is the tensor product of n such states, that is $\Psi_0(x) = (\pi \hbar)^{-n/4} e^{-|x|^2/2\hbar}$, the fiducial coherent state (1). Returning to the initial Hamiltonian H we note that the corresponding Weyl quantisation \hat{H} satisfies, in view of Eqn. (42) the symplectic covariance formula $\hat{H} = \hat{S} \hat{K} \hat{S}^{-1}$ where \hat{S} is any of the *two* metaplectic operators corresponding to the symplectic matrix S (see the Appendix). It follows that the ground state of \hat{H} is given by the formula $\Psi = \hat{S} \Psi_0$.

The case of the excited states is treated similarly. The solutions of the one-dimensional eigenfunction problem $\hat{K}_j \Psi = E \Psi$ are given by the Hermite functions

$$\Psi_N(x) = e^{-x^2/2\hbar} H_N(x/\sqrt{\hbar}) \quad (43)$$

with corresponding eigenvalues $E_N = (N + \frac{1}{2}) \hbar \omega_j$. It follows that the solutions of the n -dimensional problem $\hat{K} \Psi = E \Psi$ are the tensor products

$$\Psi_{(N)} = \Psi_{N_1} \otimes \Psi_{N_2} \otimes \cdots \otimes \Psi_{N_n} \quad (44)$$

where $(N) = (N_1, N_2, \dots, N_n)$ is a sequence of non-negative integers, and the corresponding energy level is

$$E_{(N)} = \sum_{j=1}^n (N_j + \frac{1}{2}) \hbar \omega_j. \quad (45)$$

This allows us to give a geometric description of all eigenfunctions of the generalized harmonic oscillator, corresponding to a quadratic Hamiltonian (40). We claim that:

Let Ψ be an eigenfunction of the operator

$$\hat{H} = (x, -i\hbar \nabla_x) M (x, -i\hbar \nabla_x)^T. \quad (46)$$

The symplectic capacity of the corresponding Fermi blob Ω_F is

$$c(\Omega_F) = \sum_{j=1}^n (N_j + \frac{1}{2}) h \quad (47)$$

where the numbers N_1, N_2, \dots, N_n are the non-negative integers corresponding to the state (44) of the diagonalized operator $\hat{K} = \sum_{j=1}^n \hat{K}_j$.

APPENDIX: The Metaplectic Group

The symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ has a covering group of order two, the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$. That group consists of unitary operators (the metaplectic operators) acting on $L^2(\mathbb{R}^n)$. There are several equivalent ways to describe the metaplectic operators. For our purposes the most tractable is the following: assume that $S \in \mathrm{Sp}(2n, \mathbb{R})$ has the block-matrix form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad \det B \neq 0. \quad (\text{A1})$$

The condition $\det B \neq 0$ is not very restrictive, because one shows (de Gosson [10, 16, 19], Littlejohn [27]) that every $S \in \mathrm{Sp}(2n, \mathbb{R})$ can be written (non uniquely) as the product of two symplectic matrices of the type above; moreover the symplectic matrices arising as Jacobian matrices of Hamiltonian flows determined by physical Hamiltonians of the type “kinetic energy plus potential” are of this type for almost every time t . To the matrix (A1) we associate the following quantities (de Gosson [10, 16]):

- A quadratic form

$$W(x, x') = \frac{1}{2}DB^{-1}x \cdot x - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax' \cdot x'; \quad (\text{A2})$$

the matrices DB^{-1} and $B^{-1}A$ are symmetric because S is symplectic;

- The complex number $\Delta(W) = i^m \sqrt{|\det B^{-1}|}$ where m (“Maslov index”) is chosen in the following way: $m = 0$ or 2 if $\det B^{-1} > 0$ and $m = 1$ or 3 if $\det B^{-1} < 0$.

The two metaplectic operators associated to S are then given by

$$\widehat{S}\Psi(x) = \left(\frac{1}{2\pi i\hbar}\right)^{n/2} \Delta(W) \int e^{\frac{i}{\hbar}W(x, x')} \Psi(x') d^n x'. \quad (\text{A3})$$

The fact that we have two possible choices for the Maslov index is directly related the fact that $\mathrm{Mp}(2n, \mathbb{R})$ is a two-fold covering group of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ [11, 10, 16, 9].

The main interest of the metaplectic group in quantization questions comes from the two following (related) “symplectic covariance” properties:

- Let Ψ be a square integrable function (or, more generally, a tempered distribution), and S a symplectic matrix. We have

$$W\Psi(S^{-1}z) = W(\widehat{S}\Psi)(z) \quad (\text{A4})$$

where \widehat{S} is any of the two metaplectic operators corresponding to S ;

- Let \widehat{H} be the Weyl quantisation of the symbol (= observable) H . Let S be a symplectic matrix. Then the quantisation of $K(z) = H(Sz)$ is $\widehat{K} = \widehat{S}^{-1}\widehat{H}\widehat{S}$ where \widehat{S} is again defined as above.

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